Constant-Rate Dual-Railing of the Scalar Cubic: UCNC 2025 Problem 1, First Nontrivial Case

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Abstract

The *selective dual-railing* construction of Haisler, Huang, Migunov, Mohammed, and Provence [1] converts a scalar polynomial ODE $y' = p(y)$ into a two-species chemical reaction network that tracks $y = u - v$ while keeping $u, v \geq 0$. A shared annihilation term $-k \cdot u \cdot v$ prevents the two rails from drifting apart without constraint. UCNC 2025 Problem 1 asks whether, for every bounded scalar polynomial p, there exists a *constant* k (uniform in the initial condition $y(0)$) such that the dual-railed system remains bounded for all time. We settle the first nontrivial case $p(y) = 1 - y^3$ at zero initialization: for every $k > 6$, starting from $u(0) = v(0) = 0$, the dual-railed system is bounded on $[0, \infty)$. The threshold $k = 6$ is sharp in the saddle-node sense: it is the root of $k^3 - 27k - 54 = (k-6)(k+3)^2$. The result has been formalized in Lean 4 without axioms beyond Mathlib. A companion note records numerical evidence that, starting from nonzero initialization $y(0) = U_0$, the critical rate scales as $k^* \approx 3.158 \cdot U_0$; the scaling constant appears to have no simple closed form.

1 Background: dual-railing and UCNC 2025 Problem 1

A *chemical reaction network* (CRN) is a polynomial ODE system $\dot{x} = f(x)$ whose right-hand side admits a mass-action decomposition $f_i(x) = p_i(x) - q_i(x) \cdot x_i$ with p_i, q_i nonneg on $\mathbb{R}^n_{\geq 0}$. A scalar polynomial ODE $y' = p(y)$ is *not* generally CRN-implementable, because y can take negative values and p can have negative monomials without a matching factor of y.

The *selective dual-railing* of [1] rewrites $y = u - v$ and splits $p(u-v)$ into a positive part $\hat{p}^+(u,v)$ and a negative part $\hat{p}^-(u,v)$ (both polynomials with nonneg coefficients in u, v). Adding a shared annihilation term $-k \cdot u \cdot v$, one obtains the *constant-rate dual-rail*

$$\begin{cases} u' = \hat{p}^+(u,v) - k\,uv, \\ v' = \hat{p}^-(u,v) - k\,uv, \end{cases} \qquad u, v \geq 0. \tag{1}$$

Subtraction gives $u' - v' = \hat{p}^+ - \hat{p}^- = p(u-v)$: provided the initial data match ($u(0) - v(0) = y(0)$), the difference $u - v$ tracks y exactly.

Without annihilation ($k = 0$), the system is correct but not bounded: any spurious positive drift in u is mirrored in v, so both rails blow up even when y is constant. The purpose of $k > 0$ is to consume matched pairs (u, v) and keep the rails finite.

Problem 1 (UCNC 2025, Problem 1)**.** *Let $p(y) \in \mathbb{Z}[y]$ be a polynomial whose associated ODE $y' = p(y)$ is bounded on some interval $I \subseteq \mathbb{R}$ containing* 0*. Does there exist a* constant $k > 0$ *(depending on p but not on $y(0) \in I$) such that the dual-railed system* (1) *is bounded on* $[0, \infty)$*?*

The smallest nontrivial instance is $p(y) = 1 - y^3$ with $I = [0, 1]$. This note treats the special case $y(0) = 0$ (which is also $u(0) = v(0) = 0$), the *zero-initialization no-collapse* problem.

Remark 2. *Zero initialization is important because the uniform constant k must work for every admissible $y(0)$, and in particular for $y(0) = 0$. The dual-rail variables themselves always start at zero (there is no way to distribute a positive $y(0)$ between $u(0)$ and $v(0)$ canonically without fixing a convention). Ruling out blow-up at the zero initialization is the required first step; numerical evidence (Section 9) shows it is not in general the hardest case.*

2 The scalar cubic system

For $p(y) = 1 - y^3$, the substitution $y = u - v$ and expansion give

$$1 - (u - v)^3 = 1 - u^3 + 3u^2v - 3uv^2 + v^3 = \underbrace{(1 + 3u^2v + v^3)}_{\hat{p}^+} - \underbrace{(u^3 + 3uv^2)}_{\hat{p}^-}.$$

Both $\hat{p}^\pm$ have nonneg coefficients. The dual-railed system at rate k is

$$\begin{cases} u' = 1 + 3u^2v + v^3 - k\,uv, \\ v' = u^3 + 3uv^2 - k\,uv, \end{cases} \qquad u(0) = v(0) = 0. \tag{2}$$

Theorem 3 (Main)**.** *For every $k > 6$, the system* (2) *admits a unique solution $(u, v)\colon [0, \infty) \to \mathbb{R}^2_{\geq 0}$, and*

$$\sup_{t \geq 0} \max\bigl(u(t),\, v(t)\bigr) \;\leq\; \tfrac{k}{2}.$$

The rest of this section sets up the proof; Section 4 proves the σ-drift identity, Section 6 proves the strict barrier, Section 7 concludes with Picard–Lindelöf global existence inside the invariant box.

Let $\sigma := u + v$ and $y := u - v$. Then $uv = (\sigma^2 - y^2)/4$ and, since $u(0) = v(0) = 0$, we have $\sigma(0) = y(0) = 0$.

3 Step 1: nonnegativity and dual-rail identity

Lemma 4 (Nonnegativity)**.** *Any solution of* (2) *starting at $u(0) = v(0) = 0$ satisfies $u(t), v(t) \geq 0$ on its domain of existence.*

Proof sketch. At the boundary $u = 0$, the drift is $u' = 1 + v^3 \geq 0$, and at $v = 0$ the drift is $v' = u^3 \geq 0$. Both $\hat{p}^\pm$ have nonneg coefficients and the annihilation terms vanish on the axes, so the first quadrant is forward-invariant. This is the standard CRN-implementability argument; in the Lean formalization it follows from `crn_local_nonneg` applied to the mass-action decomposition of (2). □

Lemma 5 (Dual-rail identity)**.** *With $y := u - v$, the pair $y(t)$ satisfies $y' = 1 - y^3$ with $y(0) = 0$. In particular, $0 \leq y(t) \leq 1$ for all $t \geq 0$.*

Proof. Subtracting the two equations of (2),

$$u' - v' = \hat{p}^+(u, v) - \hat{p}^-(u, v) = 1 - (u - v)^3 = 1 - y^3,$$

and the $-kuv$ terms cancel. At $y = 0$ the RHS is $1 > 0$; at $y = 1$ it is 0; between these it is positive, so y increases monotonically from 0 toward 1 and $y(t) \in [0, 1)$ for all finite t. □

4 Step 2: the σ-drift identity

Adding the two equations of (2) and using the algebraic identities $u^3 + v^3 = (u+v)^3 - 3uv(u+v)$ and $4uv = \sigma^2 - y^2$, we get a clean identity that is the hinge of the argument.

Proposition 6 (σ-drift)**.** *For any solution of* (2) *on its domain of existence,*

$$\sigma' = 1 + \sigma^3 - \frac{k}{2}\big(\sigma^2 - y^2\big). \tag{3}$$

Proof. Adding (2):

$$\sigma' = u' + v' = 1 + u^3 + v^3 + 3uv(u+v) - 2k\, uv.$$

Using $u^3 + v^3 + 3uv(u+v) = (u+v)^3 = \sigma^3$ and $2uv = (\sigma^2 - y^2)/2$ (from $4uv = \sigma^2 - y^2$):

$$\sigma' = 1 + \sigma^3 - 2k \cdot \tfrac{\sigma^2 - y^2}{4} = 1 + \sigma^3 - \tfrac{k}{2}(\sigma^2 - y^2).$$

□

In Lean, this is `scalar_cubic_sigma_drift` (`Ripple/DualRail/ScalarCubic.lean`, line 1483).

5 Step 3: the saddle-node threshold $k = 6$

Frozen-y analysis. Fix $y \in [0, 1]$ and consider the scalar ODE in σ obtained by treating y as a parameter:

$$\sigma' = Q_k(\sigma; y), \qquad Q_k(\sigma; y) := 1 + \sigma^3 - \tfrac{k}{2}\big(\sigma^2 - y^2\big). \tag{4}$$

Roots of $Q_k(\cdot; y)$ are the σ-equilibria. As k increases, two positive roots of $Q_k(\cdot; 1)$ appear via a saddle-node bifurcation.

Lemma 7 (Saddle-node)**.** *Fix $y = 1$ in* (4)*. The discriminant of the cubic $Q_k(\sigma; 1) = \sigma^3 - \frac{k}{2}\sigma^2 + \frac{k}{2} + 1$ vanishes at $k = 6$, and at $k = 6$ the polynomial has a double root at $\sigma = 2$. For $k > 6$, $Q_k(\cdot; 1)$ has two positive real roots $\sigma_-(1) < \sigma_+(1)$, both $\le k/2$.*

Proof. Write $\sigma = k\sigma/k$ and clear denominators: the cubic $2\sigma^3 - k\sigma^2 + k + 2 = 0$ has discriminant (up to positive factor)

$$\Delta(k) = k^3 - 27k - 54 = (k - 6)(k + 3)^2.$$

Thus $\Delta(6) = 0$, $\Delta(k) > 0$ for $k > 6$, $\Delta(k) < 0$ for $-3 < k < 6$. At $k = 6$: $2\sigma^3 - 6\sigma^2 + 8 = 2(\sigma - 2)^2(\sigma + 1)$, giving the double root at $\sigma = 2 = k/3$ (note $k/3$, not $k/2$). For $k > 6$ the two positive roots spread out around $\sigma = k/3$, with $\sigma_- < k/3 < \sigma_+$ and both $< k/2$. □

Remark 8. *The doubled factor $(k+3)^2$ in $\Delta(k)$ is structural, not incidental: it reflects the fact that for $k \in (-3, 6)$ the cubic $Q_k(\cdot; 1)$ has no real positive root, and as k crosses -3 a real root collides from below. Only the $k = 6$ crossing creates a barrier in the physically relevant region $\sigma \ge 0$.*

6 Step 4: the strict barrier

Lemma 7 suggests that $[0, \sigma_-(1)]$ is forward-invariant for the σ-dynamics when $|y| \le 1$. The Lean formalization uses a looser but more convenient overestimate: the half-interval $[0, k/3]$ with strict inequality at the right endpoint. Evaluating $Q_k(k/3; y)$:

$$Q_k\big(\tfrac{k}{3};\, y\big) = 1 + \tfrac{k^3}{27} - \tfrac{k}{2}\left(\tfrac{k^2}{9} - y^2\right) = 1 + \tfrac{k^3}{27} - \tfrac{k^3}{18} + \tfrac{k}{2}\, y^2.$$

The k^3-terms combine to $k^3\big(\frac{1}{27} - \frac{1}{18}\big) = -\frac{k^3}{54}$, so

$$Q_k\big(\tfrac{k}{3};\, y\big) = 1 - \tfrac{k^3}{54} + \tfrac{k}{2}\, y^2. \tag{5}$$

Lemma 9 (Strict barrier at $\sigma = k/3$)**.** *For $k > 6$ and $|y| \le 1$,*

$$Q_k\big(\tfrac{k}{3};\, y\big) \;<\; 0.$$

Proof. By (5) with $y^2 \le 1$,

$$Q_k(k/3; y) \le 1 - \tfrac{k^3}{54} + \tfrac{k}{2} = \tfrac{1}{54}\big(54 - k^3 + 27k\big) = -\tfrac{1}{54}(k-6)(k+3)^2.$$

For $k > 6$ this is strictly negative. □

Proposition 10 (Forward invariance)**.** *Let $k > 6$ and suppose $|y(t)| \le 1$ on $[0, T]$. Then any solution of the σ-ODE* (3) *with $\sigma(0) = 0$ satisfies $0 \le \sigma(t) \le k/3$ on $[0, T]$.*

Proof sketch. Upper bound. Suppose for contradiction that $\sigma(t_0) > k/3$ for some $t_0 \in (0, T]$. Let $t_1 := \inf\{t \in [0, t_0] : \sigma(t) \ge k/3\}$; by continuity $\sigma(t_1) = k/3$ and $\sigma(t) < k/3$ on $[0, t_1)$. The σ-drift at t_1 is $Q_k(k/3; y(t_1)) < 0$ by Lemma 9, so σ is strictly decreasing at t_1. Hence $\sigma(t) > \sigma(t_1) = k/3$ for t slightly below t_1, contradicting the definition of t_1.

Lower bound. At $\sigma = 0$ the drift is $Q_k(0; y) = 1 + \frac{k}{2}y^2 \ge 1 > 0$, so the lower boundary is repelling. □

In Lean this is `scalar_cubic_sigma_bound` (line 1834), with a local version on $[0, T]$ as `scalar_cubic_sigma_bou` (line 1521).

7 Step 5: Picard–Lindelöf with invariant box

The combination of nonnegativity (Lemma 4), the dual-rail identity $|y| \le 1$ (Lemma 5), and the σ-barrier (Proposition 10) constrains (u, v) to the compact box

$$\mathcal{B} \;:=\; \{(u, v) \in \mathbb{R}^2_{\ge 0} : u + v \le k/3\} \;\subseteq\; [0, k/3] \times [0, k/3].$$

Since $\mathcal{B}$ is compact and the RHS of (2) is polynomial, the RHS is Lipschitz on $\mathcal{B}$. Concretely, the cubic $u \mapsto u^3$ has Lipschitz constant $3 \cdot (k/3)^2 = k^2/3$ on a ball of radius $k/3$, and similar estimates bound the other monomials; this yields an explicit global Lipschitz constant L_k on $\mathcal{B}$.

Theorem 11 (Global existence)**.** *For every $k > 6$, the IVP* (2) *with $u(0) = v(0) = 0$ admits a unique C^1 solution on $[0, \infty)$ taking values in $\mathcal{B}$.*

Proof. Standard. Apply Picard–Lindelöf for each finite time horizon T: local existence is guaranteed by Lipschitz-on-compact, and the solution cannot leave $\mathcal{B}$ by Proposition 10 and Lemma 4. Because $\mathcal{B}$ is bounded, the solution does not explode, and the maximal interval of existence is all of $[0, \infty)$. □

The Lean formalization is `scalar_cubic_picard` (line 1850), which invokes the d-dimensional Picard wrapper from `Ripple.Core.ODEGlobal` with $\mathcal{B}$ as the invariant set. The Lipschitz estimate on the cubic monomials comes from an auxiliary lemma `cube_lipschitz_on_ball`.

Combining everything yields the top-level theorem:

Theorem 12 (= Theorem 3, Lean: `scalar_cubic_bounded`)**.** *For every $k > 6$, the IVP* (2) *admits a unique solution on $[0, \infty)$ with $\max(u(t), v(t)) \le k/3$.*

8 Lean formalization

The proof outlined above has been mechanized in Lean 4 / Mathlib.

- File: `projects/Ripple/Ripple/DualRail/ScalarCubic.lean`, 2045 lines.
- Top-level theorem: `scalar_cubic_bounded` (line 2013).
- Subordinate lemmas: `scalar_cubic_nonneg` (line 816), `scalar_cubic_dual_rail_identity` (838), `scalar_cubic_sigma_drift` (1483), `scalar_cubic_sigma_bound` (1834), `scalar_cubic_picard` (1850).
- Threshold constant: `scalarCubicThreshold := 6` (line 435).
- Sorry count: 0. Axioms used beyond Mathlib: none.
- Commit hash at closure: `b88b18b`.

The formalization closely mirrors the informal proof. The main nontrivial step was establishing the strict barrier (`scalar_cubic_sigma_bound`) without either a Lyapunov function or a Grönwall bound: the inequality $Q_k(k/3; y) < 0$ for $|y| \le 1$ is pushed through a contradiction argument using the mean-value theorem and the uniqueness of ODE solutions.

The d-dimensional Picard wrapper is stated in terms of a box-invariant bounded set rather than an abstract Lyapunov function; this matches the shape of the barrier argument and avoids having to formalize the convexity of Lyapunov sublevel sets.

9 Nonzero initialization and the universal constant K^*

The present result handles $u(0) = v(0) = 0$, corresponding to $y(0) = 0$. For UCNC 2025 Problem 1 in full, one must also handle nonzero $y(0) = U_0 \in (0, 1]$. Numerical experiments (see `notes/numerics/K_star_observations.md`) show a striking *scaling law*: starting from $u(0) = U_0$, $v(0) = 0$, the critical rate $k^*(U_0)$ above which the system remains bounded satisfies

$$\frac{k^*(U_0)}{U_0} \longrightarrow K^* \approx 3.158\,048\,614\,517\ldots \qquad \text{as } U_0 \to \infty. \tag{6}$$

Rescaling by U_0 removes the $+1$ forcing term from the dual-rail system in the limit, leaving a universal two-variable autonomous cubic in $(\tilde{\sigma}, \tilde{y}) = (\sigma/U_0, y/U_0)$:

$$\tilde{\sigma}' = \tilde{\sigma}^3 - \tfrac{K}{2}\big(\tilde{\sigma}^2 - \tilde{y}^2\big), \qquad \tilde{y}' = -\tilde{y}^3, \qquad (\tilde{\sigma}, \tilde{y})(0) = (1, 1).$$

Since $\tilde{y}' = -\tilde{y}^3$ integrates explicitly to $\tilde{y}(\tau) = (1 + 2\tau)^{-1/2}$, the $\tilde{\sigma}$-ODE becomes a non-autonomous scalar cubic with rational forcing. The critical K^* is the unique K for which the trajectory starting at $(1, 1)$ lies on the one-dimensional center manifold of the non-hyperbolic fixed point $(K/2, 0)$.

Remark 13 (On the arithmetic nature of K^*)**.** *The constant K^* does not match any obvious closed form: it differs from π by ≈ 0.016 and from $\sqrt{10}$ by ≈ 0.004, and a PSLQ search over algebraic relations up to degree 10 with coefficients bounded by 10^{15} found no relation. A closed form, if one exists, is presumably transcendental and tied to the cubic Abel-like ODE. We record this as a conjecture.*

Conjecture 14. *The universal scaling constant K^* in* (6) *has no closed form in elementary and standard special functions; equivalently, $K^* \notin \overline{\mathbb{Q}}$ and is not in the closure of $\overline{\mathbb{Q}}$ under* $\exp, \log,$ *and* Γ.

10 Open directions

1. **Quintic case.** The next polynomial in the sequence is $p(y) = 1 - y^5$ with $I = [0,1]$. The dual-rail is $\hat{p}^+ = 1 + 5u^4v + 10u^2v^3 + v^5$, $\hat{p}^- = u^5 + 10u^3v^2 + 5uv^4$. The σ-drift

$$\sigma' \;=\; 1 + \sigma^5 - \tfrac{k}{2}(\sigma^2 - y^2)$$

has critical point $\sigma^* = (k/5)^{1/3}$ (from $f'(\sigma) = \sigma(5\sigma^3 - k)$), and the saddle-node condition at $y = 1$ is the irreducible transcendental equation

$$\tfrac{3k}{10}\,(k/5)^{2/3} \;=\; \tfrac{k}{2} + 1,$$

with positive root $k \approx 13.278$. No factorization analogous to the cubic's $(k-6)(k+3)^2$ is available; we expect the threshold for every odd-degree generalization to be a specific algebraic number determined by the polynomial's Newton polygon. A scaffold of the corresponding Lean development exists in `Ripple/DualRail/ScalarQuintic.lean`, using the loose overestimate $k = 20$ as a temporary placeholder until the sharp algebraic threshold is formalized.

2. **Quintic rescaled eigenvalue.** The same rescaling procedure applied to the quintic yields the limiting ODE $\tilde{\sigma}' = \tilde{\sigma}^5 - \frac{K}{2}(\tilde{\sigma}^2 - \tilde{y}^2)$, $\tilde{y}' = -\tilde{y}^5$. The static rescaled saddle-node is $K = \sqrt{3125/27} \approx 10.758$, while the dynamic eigenvalue (critical K for which the initial point lies on the center manifold) is numerically $K^*_{(5)} \approx 5.410$. The ratio $K_{\text{static}}/K^*_{\text{dynamic}}$ grows with degree: $5.196/3.158 \approx 1.65$ for the cubic versus $10.76/5.41 \approx 1.99$ for the quintic. Whether this ratio tends to 2 (or some other clean value) as $\deg p \to \infty$ is a suggestive numerical pattern worth investigating.

3. **General scalar p.** Is there a polynomial formula expressing the UCNC 2025 constant k as a function of the coefficients of p and the bound $\sup|y|$ on I? The present proof suggests $k = \Theta(\|p\| \cdot \sup|y|^{(\deg p - 1)/\deg p})$ but we do not have a sharp statement. A clean answer would reduce UCNC 2025 Problem 1 to a quantitative discriminant estimate.

4. **Closed form for K^*.** The most intriguing question: is the constant $K^* \approx 3.158$ a period? An algebraic function of π? The non-autonomous scalar cubic $\tilde{\sigma}' = \tilde{\sigma}^3 - (K/2)\tilde{\sigma}^2 + (K/2)/(1 + 2\tau)$ does not yield to obvious substitutions; Painlevé-type classification may be the right framework.

5. **Nonzero initialization.** A formal proof of the scaling law (6), matching the numerical observation, would close UCNC 2025 Problem 1 for the scalar cubic. The present work covers only $y(0) = 0$.

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